Sets

Set Definition:

A set is any collection of objects. If a set is finite and not too large, we can describe it by listing all of its elements. For example, the equation:
\[ A = \{1, 2, 3, 4\} \]
Describes a set A made up of four elements 1, 2, 3, and 4.

\[ B = \{\text{blue, red, yellow}\} \]
Describes a set B made up of three elements blue, red, yellow.

\[ C = \{a, b, c, f\} \]
Describes a set C made up of four elements a, b, c, f.

A set is determined by its elements and not by any order in which these elements are listed, so we can define \( A = \{2, 4, 3, 1\} \).

The elements making up the set are assumed distinct and not repeated. Therefore, the set \( \{1, 2, 3, 2, 4\} \) is the same as \( \{1, 2, 3, 4\} \).

If the set is large or an infinite set, we describe it by listing the property necessary for membership:
\[ B = \{x | x \text{ is a positive, even integer}\} \]
Describes the set B made up of all positive even integer.
The vertical bar \( | \) is read *such that*, so the equation is read: **B equals the set of all x such that x is a positive, even integer**.

\[ B = \{x \mid x \text{ is a student enrolled in CS124}\} \]

\[ C = \{x \mid x \text{ is an alphabetical letter}\} = \{a, b, c, d, e, \ldots, z\} \]
Elements in X:

Given a set X, in which all elements are listed, we determine if an element a belongs to X by simply looking if a appears or not in the listing.

If the set X is described by a property, we check to see whether a satisfies the property for the elements of X. If a verifies the property we say that \( a \in X \), if a does not verify the property then we say that \( a \not\in X \).

For example if \( a = 1 \), \( a \in A \), but \( a \not\in B \)

Cardinality of a set:
If \( x \) is a finite set, we define the cardinality of \( X \), noted \( |X| \) as the number of elements in \( X \).

\(|A| = 4\),
\( B \) is not a finite set, so it does not apply

The empty set:
The set with no elements is called the empty (or null or void) set and is denoted as \( \emptyset \). \( \emptyset = \{\} \)

For example :
\( Z = \{x\mid x \) is an even number and \( x \) is an odd number\}\)
\( Z = \emptyset \), because there are no elements which satisfy this condition.
Set Equality:

Two sets $X$ and $Y$ are equal and we write $X = Y$ if $X$ and $Y$ have the same elements.

Corollary:
$X = Y \iff \forall x \in X \rightarrow x \in Y$ and $\forall x \in Y \rightarrow x \in X$

For example:
$A = \{ x \mid x^2 - x = 0 \}, B = \{0, 1\}$

$x (x+1) = 0$

$x = 0, x -1 = 0, x= 1$, so $A = B$

Subset:
Suppose that $X$ and $Y$ are sets. If every element of $X$ is an element of $Y$, we say that $X$ is a subset of $Y$ and we write $X \subseteq Y$.

For example
If $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, then $C \subseteq A$

Any set $X$ is a subset of itself, since any element in $X$ is in $X$.

If $X$ is a subset of $Y$ and $X$ does not equal $Y$, we say that $X$ is a proper subset of $Y$

Any set is a subset of itself, since any element in $X$ is in $X$.

Proper Subset:

If $X$ is a subset of $Y$ and $X$ does not equal $Y$, we say that $X$ is a proper subset of $Y$ and we write it:

$X \subset Y$

For $C = \{1, 3\}$ and $A = \{1, 2, 3, 4\}$, $C \subset A$

The empty set is a subset of every set.
Union of 2 sets:
Given two sets $X$ and $Y$, the set

$$X \cup Y = \{x \mid x \in X \text{ or } x \in Y\}$$

Is called the union of $X$ and $Y$. The union consists of all elements belonging to either $X$ or $Y$ (or both).

$A = \{1, 3, 5\}$ and $B = \{4, 5, 6\}$

$A \cup B = \{1, 3, 4, 5, 6\}$

Intersection of 2 sets:
Given two sets $X$ and $Y$, the set:

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}$$

Is called the intersection of $X$ and $Y$. The intersection consists of all elements belonging to both $X$ and $Y$.

$A \cap B = \{5\}$

The sets $X$ and $Y$ are called disjoint if $X \cap Y = \emptyset$

A and B are not disjoint.

Pairwise disjoint:
A collection of sets $S$ is said to be pairwise disjoint if whenever $X$ and $Y$ are distinct sets in $S$, $X$ and $Y$ are disjoint.

$S = \{\{1,3,4\}, \{5\}, \{2,6\}, \{7,8\}\}$ is pairwise disjoint.
The difference of 2 sets:

Given two sets X and Y, the set:

\[ X - Y = \{ x | x \in X \text{ and } x \notin Y \} \]

is called the difference (or relative complement).

The difference \( X - Y \) consists of all elements in X that are not in Y.

A- \( B = \{1, 3\} \)

B- \( A = \{4, 6\} \)

Power set:

The set of all subsets of a set X, denoted \( \wp(X) \) is called power set of X.

Example:

If \( A = \{a, b, c\} \), the members of \( \wp(A) \) are:

\( \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\} \)

All but \( \{a, b, c\} \) are proper subsets of A.

For this example
\[ |A| = 3, |\wp(A)| = 2^3 = 8 \]

Theorem:

a- If \( |X| = n \), then \( |\wp(X)| = 2^n \)

We will give a proof using mathematical induction that the power set of a set with n elements has \( 2^n \) elements.

The proof is by induction on n
Basis step:

If \( n = 0 \), \( X \) is the empty set, the only subset of the empty set is the empty set itself; thus
\[
|\mathcal{P}(X_0)| = 1 = 2^0 = 2^n
\]

Assume that (a) holds for \( n \):
For \( |X| = n \), \( |\mathcal{P}(X_n)| = 2^n \)

Let \( X_1, X_2, \ldots, X_k \), be the subsets of \( X \)

Let’s add \( \{a\} \) to \( X \), so that \( |X_{n+1}| = n + 1 \)
We form new subsets of \( X \) that include \( a \) by adding \( a \) to all the old subsets that did not include \( a \):
\[
X_1 \cup \{a\}, X_2 \cup \{a\}, \ldots, X_k \cup \{a\}
\]
The number of subsets that include \( a \) is that same as the number of subsets that do not include \( a \).

Thus the subsets of \( X_{n+1} \) are
\[
X_1, X_2, \ldots, X_k, X_1 \cup \{a\}, X_2 \cup \{a\}, \ldots, X_k \cup \{a\}
\]
And their number is \( |\mathcal{P}(X_{n+1})| = 2 \cdot |\mathcal{P}(X_n)| = 2^n \cdot 2 = 2^{n+1} \)

For example \( n=2 \) \( X_2 = \{1, 2\} \)
\[
\mathcal{P}(X_2) = \{\}, \{1\}, \{2\}, \{1, 2\}
\]

Let’s have \( n=3 \) \( X_3 = \{1, 2, 3\} \)
\[
\mathcal{P}(X_3) = \{\}, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}
\]

The universal Set:
We define the universal set \( U \) or a universe \( U \).
All sets are subsets of the Universal set $U$.

Complement:

Given a universal set $U$ and a subset $X$, the set $U - X$ is called the complement of $X$ and is noted $\overline{X}$

Let $A = \{1, 3, 5\}$ and $U = \{1, 2, 3, 4, 5\}$, then $\overline{A} = \{2, 4\}$

If $U = \{1, 2, 3, 5, 6, 7\}$, $\overline{A} = \{2, 6, 7\}$, the complement $\overline{A}$ depends on the universe $U$.

Venn Diagrams:
Venn diagrams provide pictorial view of sets.
In a Venn diagram, a rectangle depicts a universal set. Subsets of the universal set are drawn as circles. The inside of a circle represents the members of that set.
Let $U$ be a universal set and let $A, B, C$ be subsets of $U$. The following properties hold:

1- Associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

2- Commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

3- Distributive laws:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

4- Identity Laws:

$$A \cup \emptyset = A$$
$$A \cap U = A$$

5- Complement Laws:

$$A \cup \overline{A} = U$$
$$A \cap \overline{A} = \emptyset$$

6- Idempotent Laws:

$$A \cup A = A$$
$$A \cap A = A$$

7- Bound Laws:

$$A \cup U = U$$
$$A \cap \emptyset = \emptyset$$
8- Absorption Law:
\[ A \cup (A \cap B) = A \]
\[ A \cap (A \cup B) = A \]

9- Involution Law:
\[ \overline{A} = A \]

10- 0/1 Laws:
\[ \overline{U} = \emptyset \]
\[ \overline{\emptyset} = U \]

11- De Morgan’s law for sets:
\[ (A \cup B) = A \cap B \]
\[ (A \cap B) = A \cup B \]

**Proof for 6:**

By the definition of the equality of sets, we need to prove that
\[ \forall x \ [ x \in A \cap (B \cup C) \text{ if and only if } x \in (A \cap B) \cup (A \cap C) ] \]

For that we need to show that for an arbitrary element in the universe \( U \),
\[ x \in A \cap (B \cup C) \text{ if and only if } x \in (A \cap B) \cup (A \cap C) \]

Here the *only if* part is going to be proven. The *if* part can be proven
\[ x \in A \cup (B \cap C') \iff x \in A \lor x \in (B \cap C') \]
by the definition of ∪.

\[ \Leftrightarrow x \in A \lor (x \in B \land x \in C) \]
by the definition of ∩.
\[ \Leftrightarrow (x \in A \lor x \in B) \land (x \in A \lor x \in C) \]
by the distribution from the
equivalences of propositional logic.
\[ \Leftrightarrow (x \in A \cup B) \land (x \in A \cup C) \]
by the definition of ∪.
\[ \Leftrightarrow x \in (A \cup B) \cap (A \cup C) \]
by the definition of ∩.

**Proof for 8:** (a) If \( A \subseteq B \) then \( A \cup B = B \).
Let \( x \) be an arbitrary element in the universe.
\[ x \in A \cup B \Leftrightarrow x \in A \lor x \in B \]
Then
\[ A \subseteq B \Leftrightarrow x \in A \Rightarrow x \in B \]
Since
\[ x \in B \Rightarrow x \in B \]
Also
\[ x \in A \cup B \Rightarrow x \in B \]
Hence
\[ A \cup B \subseteq B \]
Hence
\[ B \subseteq A \cup B \]
Since (use "addition" rule), \( A \cup B = B \) follows.
(b) Similarly for \( A \cap B = A \).

**Alternative proof:**

These can also be proven using 8, 14, and 15. For example, (b) can be proven as follows:
First by 15 \( A \cap B \subseteq A \).
Then since \( A \subseteq A \), and \( A \subseteq B \), by 7 \( A \cap B \subseteq A \cap B \).
Since \( A \cap A = A \) by 3, \( A \subseteq A \cap B \).

**Proof for 9:** Let \( x \) be an arbitrary element in the universe.
\[ [x \in A \cup (B - A)] \Leftrightarrow [x \in A \lor (x \in B \land x \notin A)] \]
Then
\[ \Leftrightarrow [(x \in A \lor x \in B) \land (x \in A \lor x \notin A)] \]
\[ \Leftrightarrow [(x \in A \lor x \in B) \land True] \]
\[ \Leftrightarrow [x \in A \lor x \in B] \]

\[ A \cup (B - A) = A \cup B \]

Hence \( A \cap (B - A) \neq \emptyset \).

**Alternative proof**

This can also be proven using set properties as follows.

\[ A \cup (B - A) = A \cup (B \cap \overline{A}) \]
by the definition of \((B - A)\).
\[ = (A \cup B) \cap (A \cup \overline{A}) \]
by the distribution.
\[ = (A \cup B) \cap \emptyset \]
by 1.

**Proof for 10:** Suppose \( A \cap (B - A) \neq \emptyset \).

Then there is an element \( x \) that is in \( A \cap (B - A) \), i.e.
\[ x \in A \land (x \in B \land x \notin A) \]
\[ \Leftrightarrow (x \in A \land x \notin A) \land x \in B \]
\[ \Leftrightarrow \emptyset \]
Hence \( A \cap (B - A) = \emptyset \) does not hold.
Hence \( A \cap (B - A) = \emptyset \).

This can also be proven in the similar manner to 9 above.

**Proof for 11:** Let \( x \) be an arbitrary element in the universe.
\[ x \in A - (B \cup C) \Leftrightarrow x \in A \land x \notin B \cup C \]
Then
\[ \Leftrightarrow x \in A \land \neg(x \in B \lor x \in C) \]
\[ \Leftrightarrow x \in A \land (x \notin B \land x \notin C) \]
\[ \Leftrightarrow (x \in A \land x \notin B) \land (x \in A \land x \notin C) \]
\[ \Leftrightarrow x \in A - B \land x \in A - C \]
\[ \iff x \in (A \cup B) \cap (A \cup C) \]

Hence

\[ A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C) \]

**Proof for 12:**

(a) \[ B \subseteq A \] and \[ \overline{A} \subseteq B \]

Try to prove \[ \iff x \notin A \] and \[ x \in B \]

Let \( x \) be an arbitrary element in the universe.

Then if \[ x \in B \] and \[ x \notin A \]

Then \[ x \in \overline{A} \]

Hence \[ B \subseteq \overline{A} \]

If \[ x \in \overline{A} \] and \[ x \notin A \]

Then \[ x \in A \cup B \]

Since \[ x \in (A \cup B) \cup (A \cap \overline{A}) \]

must hold. Hence \[ \overline{A} \subseteq B \]

Hence \[ B = \overline{A} \]

(b) \[ \iff x \in (U \setminus A) \] and \[ x \in B \]

Since \[ B = \overline{A} \]

Also \[ A \cap B = A \cap (U \setminus A) \]

by 10 above.

**Proof for 13:**

Since \[ \overline{A} \cup A = \overline{A} \cup A \]

Also since \[ A \cap \overline{A} = \emptyset \]

Hence \( A \) satisfies the conditions for the complement of \( \overline{A} \).

Hence \[ A = \overline{A} \].