

Variables and Quantifiers

Propositional functions:

The logic we have been using so far is very limited and cannot describe most of the statements in mathematics and computer science.

For example the statement:

p : n is an odd integer is neither true nor false, because it depends on the value of n , so it is not a proposition.

If a statement truth value depends on the value of something, usually a name, in the statement, then there is a need to express that statement in terms of a *variable*.

Definition:

Let $P(x)$ be a statement involving the variable x and let D be a set. We call P a propositional function (with respect to D) if for each x in D , $P(x)$ is a proposition.

We call D *the domain of discourse* or domain of P .

Let $P(n)$ be the statement :

n is an odd integer

and let D be the set of positive integers.

Then P is a propositional function with the domain D , since for each n in D , $P(n)$ is true or false but not both.

For example:

if $n = 1$, we obtain the proposition:

1 is an odd integer (True).

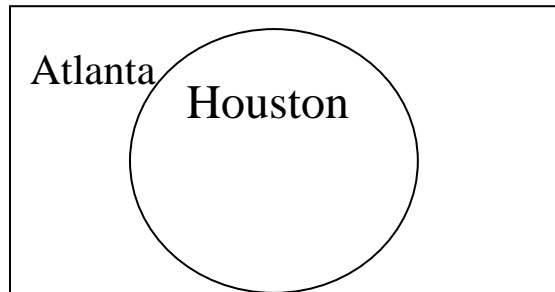
If $n = 2$, we obtain the proposition

2 is an odd integer (False)

A propositional function is true or false of an individual.

For example:

$p(x)$: x is in Texas, D = set of cities and towns in the US.
 $p(x)$ is true of Houston; it is not true of Atlanta.



$p(n)$: $n^2 + 2n$ is an odd integer, (D = set of positive integers)

For each positive integer value of n , we obtain a proposition.

For example for :

$n=1$, $p(1) = 2$ and $p(1)$ is true

$n=2$, $p(2) = 8$ and $p(2)$ is false.

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Propositional functions are used to classify things (is it true of them or not)

1. What in particular is it true or false of?
2. How many things is it true or false of?

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Quantifiers and how many:

A quantifier is an answer to a "how many" question: so is the proposition true for some, for all, no one, a few, not everyone etc..

For all, some, no one, not everyone are called **quantifiers**.

For example:

Which of the following quantified sentences correspond to the

sentence: John is sleeping through the class.

1. Someone is sleeping through the class
2. Everyone is sleeping through the class
3. No one is sleeping through the class
4. Not everyone is sleeping through the class

The answer is 1, that means that the proposition is true for some people, (at least one)

These quantifiers have symbolical representations (to learn)

Also, most of the statements in mathematics and computer science use terms such as “for every” , “for some” , etc...

For example :

For every triangle T, the sum of the angles of T is equal to 180°

Definition:

Let P be a propositional function with domain of discourse D, the statement:

for every x, P(x)

is said to be a **universally quantified statement**.

This statement may be written:

$\forall x, p(x)$

The symbol \forall is called a *universal quantifier*.

\forall means **for all**

Truth-value of a for all statement:

The statement: $\forall x, p(x)$ is true if p(x) is true **for every** x in D.

The statement $\forall x, p(x)$ is false if p(x) is false **for at least one** x in D.

For example:

$\forall x, p(x)$: For all x , x has four limbs.

D = set of mammals

For every animal x in the set of mammals, $p(x)$ is **true**.

To prove this, we need to show that it is true for every x in the set D .

$\forall x, p(x)$: For all x , x has four limbs. (True for all x in D)

D_1 = set of vertebrates.

For every animal x in the set of vertebrates, $p(x)$ is false, because fish is a vertebrate that does not have any limbs.

$\forall x, p(x) = x^2 + 2x$ is an odd integer

D = set of positive integers

is false because for $x=2$, $p(2) = 8$, 8 is an even number so $p(x)$ is false.

Existential Quantifier:

The statement:

For some x , $p(x)$ is said to be an **existentially quantified** statement.

The symbol \exists means for some, or there exists, thus the statement:

for some x , $p(x)$ may be written: $\exists x, p(x)$

The symbol \exists is called **existential quantifier**.

Truth Value for a *for some* statement:

The statement

$\exists x, p(x)$ is true if $p(x)$ is true for **at least one** x in D

The statement

$\exists x, p(x)$ is false if $p(x)$ is false **for every** x in D .

For example:

1. $\exists x, p(x)$: for some x , x has four limbs.

D = set of all vertebrates

Let $x = \text{cat}$, $p(\text{cat})$: cat has four limbs.

$p(\text{cat})$ is true, so the proposition

$\exists x, p(x)$: for some x , x has four limbs is **true**.

2. $\exists x, p(x)$: for some x , x has four limbs.

D = set of worms

$\exists x, p(x)$: for some x , x has four limbs is **false**

To prove this, we have to prove that for every element x in the set D , $p(x)$ is false.

Proving universally quantified statements:

Proving that a universally quantified statement is true means proving that **it is true for every x in the domain of discourse**.

For every real number x , $x^2 \geq 0$ is true is a universally quantified statement.

The domain of discourse \mathfrak{R} is the set of real numbers.

Proof:

We'll consider the two cases of value of x :

1. Let x be a positive real number

$x^2 = x \cdot x$, the product of two positive numbers is a positive number.

We have proven that for $x > 0$ x^2 is positive.

2. Let x be a negative real number

x can be re-written as $(-1)|x|$ where $|x|$ is the absolute value of x

$$x^2 = (-1) \cdot |x| \cdot (-1)|x| = (-1)(-1)|x||x| = 1|x|^2 > 0$$

We have proven that for $x < 0$ x^2 is positive.

The propositional function $p(x) > 0$ for every real number x is then true.

Prove that

\forall real number x , if $x > 1$, then $x+1 > 1$

We must verify that this true for every real number x that is greater than 1.

Let x be a real number such that $x > 1$.

It is true that for every real number:

If $x > 1$, then by adding 1 to both sides of the inequality, we obtain:

$x+1 > 1+1$. For this, we obtain that:

$x+1 > 2$, since $2 > 1$, we can conclude that: $x+1 > 1$

So if $x > 1$, then $x+1 > 1$ is true

Proving that a universally quantified statement is **false** means

proving that there exist an element x for which the statement is false.

Prove that:

For every real number x , $x^2 - 1 > 0$ is false.

\forall real number x , $x^2 - 1 > 0$ is false

To prove that a universally quantified statement is false in a domain of real numbers, we have to find at least one x for which the statement is false.

For $x=1$, we replace x by its value (1) and we obtain:

$$(1)^2 - 1 = 1 - 1 = 0,$$

0 is not greater than 0, so the statement

$\forall x$ real, $x^2 - 1 > 0$ is false

Proving existentially quantified statements:

To prove that an **existentially** quantified statement is **true**, we have to **find one element x for which the statement is true.**

For example:

For some positive integer n , if n is prime, then $n+1$, $n+2$, $n+3$, $n+4$ are not prime.

Symbolically we can write this as:

$\exists n$ integer, if n is prime that $n+1$, $n+2$, $n+3$, $n+3$ not prime.

Let's find an integer for which this is true:

$n = 23$ is a prime number, $n+1 = 24$, $n+2 = 25$, $n+3 = 26$ and $n+4 = 27$ are not prime numbers.

We have proven that

\exists n integer, if n is prime that n+1, n+2, n+3, n+3 not prime.

To prove that an **existentially** quantified statement is **false** in D, **we have to prove that it is false for every x in D.**

Let's prove that:

For some real number x, $\frac{1}{x^2 + 1} > 1$ is false.

We must show that $\frac{1}{x^2 + 1} > 1$ is false **for every** real number x.

$\frac{1}{x^2 + 1} > 1$ is false precisely when $\frac{1}{x^2 + 1} \leq 1$ is true.

Thus, we must show that $\frac{1}{x^2 + 1} \leq 1$ is true for every real number x.

Let x be a real number whatsoever, x^2 is a positive number, so: $0 \leq x^2$, we add 1 on both sides of the inequality to obtain:

$$1 \leq x^2 + 1$$

If we divide both sides of the inequality by $x^2 + 1$, the direction of the inequality does not change since $x^2 + 1$ is a positive number, we obtain:

$$\frac{1}{x^2 + 1} \leq 1$$

Therefore the statement

$\frac{1}{x^2 + 1} \leq 1$ is true for every real number x .

Thus, its negation: $\frac{1}{x^2 + 1} > 1$ is false for every real number x .

If the proposition is false for every real number, then it is also false for some real numbers.

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Generalized De Morgan Law for Logic (to learn):

If P is a propositional function, each pair of proposition in (a) and (b) have the same truth value

$$(a) \neg (\forall x, p(x)) \Leftrightarrow \exists x, \neg p(x)$$

$$(b) \neg (\exists x, p(x)) \Leftrightarrow \forall x, \neg p(x)$$

Proof:

Suppose that :

$\neg(\forall x, p(x))$ is true. Then the proposition $\forall x, p(x)$ is false.

By definition, the proposition $(\forall x, p(x))$ is false when $p(x)$ is false for at least one x in the domain of discourse.

So there exist one x for which $p(x)$ is false.

But if $p(x)$ is false for at least one x in the domain of discourse, the $\neg p(x)$ is true for at least one x in the domain of discourse.

Again by definition, when $\neg p(x)$ is true for at least one x in the

domain of discourse, the proposition

$\exists x, \neg p(x)$ is true

Thus if the proposition :

$\forall x, \neg p(x)$ is true, the proposition $\exists x, \neg p(x)$ is also true.

Similarly

if $\neg(\forall x, p(x))$ is false, the proposition $\exists x, \neg p(x)$ is also false, so the pair of propositions in (a) always has the same truth-value.

Let's consider the compound proposition:

$$P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n$$

The proposition $P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n$ is true if and only if P_i is true for every $i = 1, 2, \dots, n$.

It is false if there is one P_i for which P_i is false.

A universally quantified proposition generalizes the compound proposition:

$$P_1 \wedge P_2 \wedge P_3 \wedge \dots \wedge P_n$$

in the sense that the individual propositions $P_1, P_2, P_3, \dots, P_n$ are replaced by a family $P(x)$, where x is a member of the domain of discourse and is written as for every $x, P(x)$

The truth value of the proposition for every $x, P(x)$ is defined similarly, that is $P(x)$ is true if and only if $P(x)$ is true for every x in the domain of discourse.

Similarly, an existentially quantified proposition

$$P_1 \vee P_2 \vee P_3 \vee \dots \vee P_n$$

Is true if only one P_i is true and is false if all P_i are false for $i=1, \dots, n$.

An existentially quantified proposition $\exists x, P(x)$ generalizes the compound proposition:

$$P_1 \vee P_2 \vee P_3 \vee \dots \vee P_n$$

in the sense that the individual propositions $P_1, P_2, P_3, \dots, P_n$ are replaced by an family $P(x)$, where x is a member of the domain of discourse and is written as for some $x, P(x)$

The truth value of the proposition for some $x, P(x)$ is defined similarly, that is $P(x)$ is true if there is one x such that $P(x)$ is true and is false if and only if $P(x)$ is false for every x in the domain of discourse.

The preceding observations on the first DeMorgan Laws of Logic that the propositions:

$$1. \neg(p_1 \vee p_2 \vee \dots \vee p_n) \equiv \neg p_1 \wedge \neg p_2 \wedge \dots \wedge \neg p_n$$

We replace each side of the equivalence by its counterpart and we get:

$$\neg(\exists x, P(x)) \equiv \forall x \neg P(x)$$

Translating from English into Logical Expressions

Translating sentences in Natural Language (as English) into logical expressions is an essential part of specifying both hardware and software systems.

System and software engineers take requirements in natural language and produce precise, unambiguous specifications that can be used as the basis for system development.

Express the statement:

“The automated reply cannot be sent when the file system is full”

One way to represent this is to define a single proposition S : “The automated reply cannot be sent when the file system is full”

This is the simplest and least useful way to represent since we cannot analyze the sentence or reason about it.

A better way to translate this is to let p denote “The automated reply can be sent” and q denote: “the file system is full”. Then $\neg p$ represents: “It is not the case that the automated reply can be sent”, which also means “The automated reply cannot be sent”. Consequently, our specification can be expressed as:

$$q \rightarrow \neg p$$

Every student in this class has studied calculus

Using variables and quantifiers:

1. Rewrite the statement so that we can identify the appropriate quantifier to use:

For every student in this class, that student has studied calculus

2. Introduce a variable x , so that our statement becomes:
For every student x in this class, x has studied calculus.

3. Introduce a proposition $C(x)$, which is the statement: “ x has studied calculus”.

4. Choose the universe of discourse:

If the universe of discourse consists of the student in the class, the statement is expressed as:

$$\forall x, C(x).$$

If we change the domain of discourse to consist of all people, we need to express our statement as:

“For every person x , if person x is a student in this class then x has studied calculus”

if $S(x)$ represents the statement:

“ x is a student in this class”,

our statement can be expressed as:

$$\forall x (S(x) \rightarrow C(x))$$

Caution: we cannot express our statement as:

$\forall x (S(x) \wedge C(x))$, since this statement says that all people are students in this class and have studied calculus.

- All students are happy:

$S(x)$: x is a student

$H(x)$: x is happy

$$\forall x, S(x) \rightarrow H(x)$$

- Some students are majoring in CS

If we let the proposition $P(x)$: “ x is student”, and $Q(x)$ be “ x is majoring in CS”

This can be re-written as:

$$\exists x P(x) \wedge Q(x)$$

- Some apples are red

If we let the proposition $P(x)$: "x is an apple", and $Q(x)$ be "x is red"

This can be re-written as:

$$\exists x (P(x) \wedge Q(x))$$

- All humans are mortal:

If we let $P(x)$ be the proposition "x is human", and $Q(x)$ the proposition "x is mortal"

This can be re-written as:

$$\forall x, P(x) \rightarrow Q(x)$$

All senior citizens and students get a discount:

$P(x)$: x is a senior citizen

$Q(x)$: x is a student

$S(x)$: x gets a discount.

$$\forall x, P(x) \vee Q(x) \rightarrow S(x)$$

All students like Logic

$S(x)$: x is a student

$L(x)$: x likes logic

$$\forall x, S(x) \rightarrow L(x)$$

If a person is female and is a parent, then this person is someone's mother:

For every person x, if person x is female and person x is a parent, then there exists a person y such that x is the mother of y.

We introduce:

$F(x)$: x is female

$P(x)$: x is parent

$M(x, y)$: x is mother of y

The original statement is represented as:

$$\forall x((F(x) \wedge P(x)) \rightarrow \exists y M(x, y))$$

we can move $\exists y$ to the left of the condition because y does not appear in $F(x) \wedge P(x)$, and we obtain an equivalent expression:

$$\forall x \exists y ((F(x) \wedge P(x)) \rightarrow M(x, y))$$

All mothers are female

$F(x)$: x is female

$M(x,y)$: x is mother of y

$$\forall y \exists x (M(x, y) \rightarrow F(x))$$

if x and y are siblings, x is either the brother of y or x is the sister of y .

$S(x, y)$: x is sibling of y

$B(x, y)$: x is brother of y

$C(x,y)$: x is sister of y

$$\forall x, \forall y, S(x, y) \rightarrow B(x, y) \vee C(x,y).$$

Every dog has a tail.

$D(x)$: x is a dog

$H(x,y)$: x has a tail y

$$\forall x \exists y (D(x) \rightarrow H(x,y))$$

Nested Quantifiers:

Nested quantifiers are quantifiers that occur in the scope of other quantifiers, such as the statement:

$$\forall x \exists y (x+y = 0)$$

Assume that the universe of discourse for the variables x and y consists of all real numbers.

The statement:

$$\forall x \forall y (x+y = y+x)$$

says that

$x+y = y+x$ for all real numbers x and y .

The statement

$\forall x \exists y (x+y = 0)$ says that:

For every real number x there is a real number y such that $x+y = 0$.

Translate into English:

$$\forall x \forall y ((x > 0) \wedge (y < 0) \rightarrow (x \cdot y < 0))$$