Proofs

A mathematical system consists of axioms, definitions and undefined terms.
An axiom is assumed ***true***.
Definitions are used to create new concepts in terms of existing ones.
Undefined terms are only defined implicitly defined by the axioms.
Within a mathematical system, we can derive theorems.
A theorem is a proposition that has been proved true.
A lemma is a special kind of theorem that is not usually interesting.
A corollary is a theorem that follows quickly from another theorem.
Euclidian geometry is an example of mathematical system.

Example of an axiom in this system:
Given two distinct points, there is exactly one line that contains them.

The terms points and line are undefined terms that are implicitly defined by the axiom.

Example of a definition:

Two angles are supplementary if the sum of their measure is 180°.

If two sides of a triangle are equal, then the angles opposite them are equal.

A corollary is:
"If a triangle is equilateral, then it is equiangular."

Theorems are often of the form:

Theorem 1:
for all $x_1, x_2, \ldots, x_n$, if $p(x_1, x_2, \ldots, x_n)$ then $q(x_1, x_2, \ldots, x_n)$

This universally quantified statement is true provided that the conditional statement:
if $p(x_1, x_2, \ldots, x_n)$ then $q(x_1, x_2, \ldots, x_n)$
is true for all $x_1, x_2, \ldots, x_n$.

Direct proof:

To prove that Theorem 1 is true, we assume that:

1. $x_1, x_2, \ldots, x_n$ are arbitrary members of the domain of discourse.
2. We also assume that $p(x_1, x_2, \ldots, x_n)$ is true
3. then , using $p(x_1, x_2, \ldots, x_n)$ as well as other axioms, definitions and previously derived theorems, we show directly that $q(x_1, x_2, \ldots, x_n)$ is true

We will give a direct proof for the following statement:

For all real numbers d, d_1, d_2, and x
If $d = \min \{d_1, d_2\}$ and $x \leq d$ then $x \leq d_1$ and $x \leq d_2$

Proof:

We will assume that d, d_1, d_2 and x are arbitrary real numbers.
We will assume that
{$d = \min \{d_1, d_2\}$ and $x \leq d$ }is true and then prove that
: $x \leq d_1$ and $x \leq d_2$

From the definition of a minimum, it follows that :
$d \leq d_1$
and
$d \leq d_2$

From $x \leq d$ and $d \leq d_1$, we derive that $x \leq d_1$

From $x \leq d$ and $d \leq d_2$, we derive that $x \leq d_2$,
Therefore, $x \leq d_1$ and $x \leq d_2$

Indirect Proof:

Since the implication $p \rightarrow q$ is equivalent to its contrapositive $\neg q \rightarrow \neg p$, the implication $p \rightarrow q$ can be shown by proving its contrapositive $\neg q \rightarrow \neg p$ is true.

Prove that
For all real number x and y, if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$.

Proof:
Suppose that:
We assume the conclusion is false:
  either $x \geq 1$ or $y \geq 1$ is false

So, its negation : $\neg$ (either $x \geq 1$ or $y \geq 1$) is True,

- And should conclude that the hypothesis is false : $x+y \geq 2$ is false

$\neg$ (either $x \geq 1$ or $y \geq 1$) is True $\Leftrightarrow$ $x < 1$ and $y < 1$.
Note that negating or results in and.
  $x < 1$ and $y < 1$
We add each corresponding member of the inequality and we get:
  $x+y < 1+1 = 2 \Leftrightarrow x + y < 2$
At this point, we have derived $\neg p$
where $p : x+y \geq 2$.
Thus we conclude that the statement:

For all real number x and y, if $x+y \geq 2$, then either $x \geq 1$ or $y \geq 1$ is true.

This special case of proof is called a *proof by contrapositive*.

Deductive reasoning:

The process of drawing a conclusion from a sequence of propositions is called deductive reasoning.
For example:

1. The bug is either in module 17 or in module 81.
2. The bug is a numerical error
3. Module 81 has no numerical error.

If these statements are true, it is reasonable to conclude that

The bug is in module 17.

The given propositions are called hypotheses, and the proposition that follows from the hypotheses is called conclusion

Valid argument:
Any argument of the form
If p_1 and p_2 and …p_n then q can be written as a sequence of propositions:

p_1

p_2

.

.

p_n

$\therefore q$

The propositions p_1 p_2 …p_n are called the premises or the hypotheses and q is called the conclusion.
The argument is valid provided that:
if p_1 and p_2 … and p_n are all true, then q must also be true .Otherwise, the argument is invalid or a fallacy.

Determine whether

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

is valid

Proof :
Suppose that $p \rightarrow q$ and p is true. Then q must be true, otherwise $p \rightarrow q$ would be false.
Therefore, the argument is valid.

Rules of inference for propositions:

For a proof as a whole to be valid, each step of the proof has to draw intermediary conclusions that are valid.

To make sure that the intermediary conclusions are valid, each conclusion has to be deduced using some of the rules of inference listed below:

Rule of inference	Name
$p \to q$ p $\therefore q$	Modus Ponens
$p \to q$ $\neg q$ $\therefore \neg p$	Modus Tollens
p $\therefore p \vee q$	Addition
$p \wedge q$ $\therefore p$	Simplification
p q $\therefore p \wedge q$	Conjunction
$p \to q$ $q \to r$ $\therefore p \to r$	Hypothetical syllogism

$p \vee q$ $\neg p$ $\therefore q$	Disjunctive syllogism

Which rule of inference is used in the following argument:

If the computer has 32 Meg of RAM, then it can run “SoundBlaster”, if it can run “SoundBlaster” then the sonics will be impressive.
Therefore, if the computer has 32 Meg then the sonics will be impressive.
P : “The computer has 32 meg of RAM”
Q : “The computer can run “SoundBlaster””
R : “The sonics will be impressive”

The arguments can be represented as:

$$\begin{array}{l} P \rightarrow Q \\ Q \rightarrow R \\ \hline \therefore P \rightarrow R \end{array}$$

This argument uses the hypothetical syllogism rule, so it is true.

Let’s have the following argument:

if I study hard, then I get A’s
I study hard.
Let p: “I study hard”, q: I get A’s
If I study hard then I get A’s
I study hard

∴ I get A’s

$p \rightarrow q$
p

$\therefore q$

This argument uses the modus ponens, therefore it is valid.

Let’s have the argument:

If I study hard, then I get A’s,
I do not get A’s, therefore I do not study hard.

$p \rightarrow q$
$\neg q$

$\therefore \neg p$

This argument uses the modus tollens, therefore it is valid.

Let’s have the argument:

I study hard, therefore I study hard or I will get rich

$$\frac{p}{\therefore p \vee q}$$

This argument uses the addition rule of inference, therefore it is valid.

Let's have the argument:

I study hard and I get A's, therefore I study hard

$$\frac{p \wedge q}{\therefore p}$$

This argument uses the simplification rule of inference, therefore it is valid

Let's have the argument:
I study hard, I have A's, therefore I study hard and I have A's

$$\frac{\begin{array}{c} p \\ q \end{array}}{\therefore\ p \wedge q}$$

This argument uses the conjunction rule of inference therefore it is valid

Let's have the argument:

I study hard or I get A's, I do not get A's, therefore I study hard

$p \vee q$
$\neg q$

$\therefore p$

This argument uses the disjunctive syllogism rule of inference therefore it is valid

Rules of Inference for Quantified Statements:

There are 4 rules of inference for quantified statements:
Let x be a universally quantified variable in the domain of discourse D, such that p(x) is true,

$\forall x \in D, p(x)$ is true

Rule of Inference	Name
$\forall x \in D\ P(x)$ -------------------- $\therefore$ P(d) if $d \in D$	Universal instantiation
P(d) for any $d \in D$ --------------------------- $\forall x \in D\ P(x)$	Universal generalization
 $\exists x \in D\ P(x)$ -------------------- $\therefore$ P(d) for some $d \in D$	Existential instantiation
P(d) for some $d \in D$ --------------------------- $\exists x \in D\ P(x)$	Existential generalization

Everyone who is taking CS124 wants an A.

Let's D be the domain of all students taking CS124
P(x) : x wants an A
$\forall x \in D\ P(x)$
John $\in$ D, therefore, P(john) is true. (universal instantiation)

P(d) for any d in D , for any student d in CS124,
P(d)= "d wants an A" is true,

$\forall x \in D\ P(x)$

Some students taking CS124 love proofs

Let's D be the domain of all students taking CS124

Let Q denote the proposition: x loves proofs
We can re-write this as :
$\exists x \in D\ Q(x)$

This means that we can find at least one student d in CS124 for which Q(d) is true.

Conversely, if we can find at least one student d in CS124 for which Q(d) is true, then $\exists x \in D\ Q(x)$ is true.

Combining propositional and quantified statement inferences:

Let's have:

1- everyone loves either MS-DOS or UNIX.
2- Lynn does not love MS-DOS.

Show that the conclusion "Lynn loves UNIX" follows from the hypotheses

Let p(x) denote the propositional function: "x loves MS-DOS"
Let q(x) denote the propositional function: "x loves UNIX"

The first hypothesis can be re-written as:
$\forall x\ p(x) \vee q(x)$
By universal instantiation, we have:
$p(Lynn) \vee q(Lynn)$

The second hypothesis is $\neg$ p(Lynn)

p(Lynn) $\vee$ q(Lynn)
$\neg$ p(Lynn)
----
q (Lynn) by disjunctive syllogism.

Therefore, the conclusion follows from the hypotheses.

Quantifiers and Logical Operators

1. $\forall$ x [p(x) $\wedge$ q(x)] $\Leftrightarrow$ [$\forall$ x p(x) $\wedge$ $\forall$ x q(x)]
2. $\forall$ x [p(x) $\vee$ q(x)] $\Leftrightarrow$ [$\forall$ x p(x) $\vee$ $\forall$ x q(x)]
3. $\exists$ x [p (x) $\wedge$ q(x)] $\Leftrightarrow$ [$\exists$ x p(x) $\wedge$ $\exists$ x q(x)]
4. $\exists$ x [p (x) $\vee$ q(x)] $\Leftrightarrow$ [$\exists$ x p(x) $\vee$ $\exists$ x q(x)]